# Stability of weakly nonlinear localized states in attractive potentials

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We analyze the stability of bound states to the nonlinear Schrödinger equation with an "attractive" linear potential and a cubic nonlinearity of arbitrary sign. A sufficient stability criterion is derived, which only requires knowledge of the linear modes of the potential. The results are double-checked numerically for the step-index optical fiber. An estimate of the growth rate versus nonlinearity is established in the limit of weak power.

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## I. INTRODUCTION

The detailed understanding of nonlinear effects in optical systems has been the goal of many research activities in recent years [1]. With the rapid development of both powerful and controllable light sources, many challenging effects were discovered just by increasing the intensity. For example, by enhancing the intensity of the optical field in a waveguide, self-focusing due to the optical Kerr effect can change the guiding properties dramatically. This self-focusing process is basically described by the nonlinear Schrödinger (NLS) equation, which governs the evolution of the slowly varying envelope of the electric field, and it can partly be "tamed" by coupling the beam with an appropriate potential [2–4].

Besides nonlinear optics, elementary excitations in cold dilute atom gases and the formation of Bose-Einstein condensates (BEC's) have attracted interest in the past decade [5]. In the mean-field approximation, the dynamics of BEC's is described by the Gross-Pitaevskii equation, which governs the macroscopic BEC wave function. This equation is nothing else but a NLS equation supplemented by an unbounded quadratic-in-space potential. Here, the potential models the magnetic trap, confining bosons into a condensate. For atoms with attractive interactions, BEC's can undergo sequences of collapses [6], similar to the self-focusing phenomenon in optics. However, for suitable numbers of particles and/or different interactions, long-living stationary-wave structures such as ground states (single humped) or vortices (with angular momentum; see, e.g., [7]) can form in the condensates.

The above systems promote the emergence of a rich variety of nonlinear objects, the stability of which crucially depends on the trap potential and the number of quanta involved in their formation (see for example [8] for one-dimensional systems). As far as the physics of BEC's is concerned, recent studies focused on the stability of ground states as well as vortices by means of perturbation theory (see, e.g., Refs. [9,10]). The stability of similar trapped structures was also investigated in the framework of nonlinear optics in, e.g., [2,3]. In particular, it was observed that single (unit) vortices with small enough power could be stable in a parabolic trap and preserve their radial shape, apart from an azimuthal rotation [3].

In spite of these studies, we believe that a simple, tractable criterion for the stability of higher-order nonlinear bound states for NLS-type systems involving a trap (or an "attractive" potential) is still missing. By higher-order bound states, we mean stationary-wave solutions of the NLS equation, with a finite power above that of the unique, positive, and symmetric (localized) ground state with the lowest power and azimuthal zero eigenvalue. Single-charged as well as multicharged vortices and multihumped field distributions belong to this class of stationary-wave solutions, which is investigated here.

In this paper, we present an easy-to-use *sufficient* stability criterion for low-power nonlinear bound states of NLS systems with an "attractive" potential. Knowledge of the spectrum of this potential is sufficient to determine the stability of the nonlinear bound states in the limit of small powers. Due to the generality of the equation under consideration, our results are applicable on both weakly nonlinear guided waves in optical fibers and the dynamics of BEC's. Moreover, the theory can be applied in principle to conservative, Hamiltonian systems of arbitrary dimensionality and an arbitrary shaped, but "attractive" potential. However, for the sake of conciseness, we mainly concentrate on the case of a circularly symmetric fiber with focusing or defocusing cubic nonlinearity.

The paper is organized as follows: In Sec. II, we determine the linear modes of the potential and their continuation towards nonlinear bound states ("solitons") for higher-power levels. In Sec. III, we discuss the stability of these "solitons" for low powers. Here we use the key property that the linear modes of the potential are related to the eigenfunctions of the linearized NLS operator, which determines the stability of weakly nonlinear waves. More precisely, each of the linear modes appears twice as an eigenfunction, but with shifted eigenvalue. Whereas it is known that the resonance of two linear *localized* eigenfunctions can produce instabilities (see, e.g., Refs. [10–12]), the present theory, instead, deals with resonances between *localized* eigenfunctions (discrete eigenvalues) and *delocalized* eigenfunctions (continuous eigenvalues). In Sec. IV, we propose a stability criterion for low-

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power nonlinear bound states, deduced from the existence of the previous resonances. By analyzing the dependence of the perturbation amplitudes on the growth rate, it is shown that the criterion is indeed sufficient for stability. Finally, in Sec. V our analytical arguments are double-checked by means of a numerical stability analysis and by direct numerical simulations.

## II. LINEAR MODES AND NONLINEAR BOUND STATES

To start with, let us consider the propagation equation for the field envelope $E(r,\varphi,z)$ in an optical waveguide with refractive index distribution $n(r)$, $\lim_{r\to\infty} n(r)=n_b$, and a Kerr nonlinearity [1]:

$$i\frac{\partial}{\partial z}E + \frac{1}{2k_0}\Delta_\perp E + k_0\frac{n_2|E|^2}{n_b}E + k_0\frac{n-n_b}{n_b}E = 0. \qquad (1)$$

For convenience, we have scaled the field $E$ in such a way that $|E|^2$ corresponds to the optical intensity. Assuming that both the linear and nonlinear induced index changes $n-n_b$ and $n_2|E|^2$ are small compared to the mean index $n_b$, the scalar approximation and the negligence of backward running field components are justified. In Eq. (1), $k_0$ refers to the carrier central wave number in the medium.

For technical convenience, we rescale Eq. (1) to dimensionless quantities. The transverse coordinate $r$ is scaled to the extension of the waveguide $r_0$, and we normalize the coefficients in front of both diffraction term and nonlinearity to unity. With $R=r/r_0$, $\Phi=\varphi$, $Z=z/2k_0r_0^2$, and $\sigma=\mathrm{sgn}(n_2)$ the two-dimensional (2D) NLS equation for the wave function $\Psi=k_0r_0(\sqrt{2|n_2|/n_b})E$ reads

$$i\frac{\partial}{\partial Z}\Psi + \Delta_\perp\Psi + \sigma|\Psi|^2\Psi - V\Psi = 0, \qquad (2)$$

with an "attractive" bounded potential $V=2k_0^2r_0^2(n_b-n)/n_b$, satisfying $\lim_{R\to\infty}V=0$.

The potential $V$ is assumed to support several localized linear modes $\Theta_{j,M}(R)\exp(iM\Phi+i\gamma_{j,M}Z)$, $M=\ldots,-1,0,1,\ldots$, $j=1,2,\ldots$, with discrete eigenvalues $\gamma_{j,M}$ ordered as $\gamma_{j_1,M}>\gamma_{j_2,M}\Leftrightarrow j_1<j_2$. The eigenfunctions $\Theta\equiv\Theta_{j,M}$ obey

$$\gamma\Theta = \hat{D}_M\Theta,$$

$$\hat{D}_M = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) - \frac{M^2}{R^2} - V, \qquad (3)$$

where $|\min V|>\gamma_{j,M}>0$. For $\gamma=\gamma_c\leqslant 0$ in Eq. (3) we have a continuum of delocalized (radiative) modes $\Theta_{\gamma_c,M}$. Further on, we consider a class of potentials $V$, for which the discrete spectrum is not degenerated, apart from the trivial degeneration that follows from the elementary symmetry $M\to -M$.

In the following, we use a radial step potential [$V(R\leqslant 1)=-V_0$, $V(R>1)=0$] as an illustrative example, which models the most common case of an optical fiber. The depth of the potential $V_0$ determines the number and magnitude of the discrete eigenvalues $\gamma_{j,M}$ and the domain of the continuum. Figure 1 shows an example for the discrete (localized modes) and continuous (delocalized modes) eigenvalues of Eq. (3).

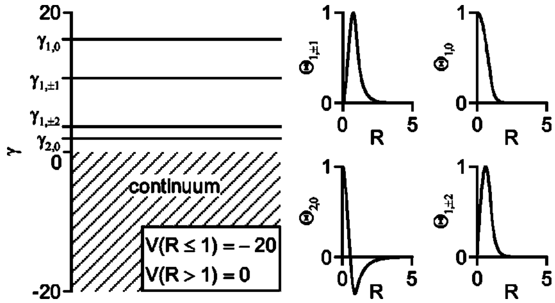


FIG. 1. Discrete eigenvalues $\gamma_{j,M}$ of Eq. (3) and radial shapes of the corresponding localized linear modes $\Theta_{j,M}$. The given step potential supports four localized modes.

From each of the localized modes $\Theta_{j,M}$ a branch of nonlinear bound states $U_{j,M}(R)\exp(iM\Phi+i\beta_{j,M}Z)$ of Eq. (2) emanates [3,13,14]. One can interpret the localized linear modes $\Theta_{j,M}$ as "solitons" with zero power $P=2\pi\int|U_{j,M}|^2RdR$ or, equivalently, corresponding to $\sigma=0$ (see Fig. 2). Because both $U$ and $\Theta$ are solutions of real-valued differential equations, we consider each of them as being real and therefore unique.

By means of functional relations (see, e.g., Ref. [3]), nonlinear bound states $U\equiv U_{j,M}$ of Eq. (2) can easily be proved to satisfy

$$\beta = \frac{\sigma\int U^4RdR - \int(\nabla U)^2RdR - \int U^2VRdR}{\int U^2RdR}, \qquad (4)$$

where $(\nabla U)^2=[(\partial/\partial R)U]^2+(M^2/R^2)U^2$ and $\beta\equiv\beta_{j,M}$. One sees immediately that for $\sigma>0$ we can expect $\beta>\gamma_{j,M}$ and $\sigma<0$ indicates $\beta<\gamma_{j,M}$.

## III. STABILITY ANALYSIS

According to the standard procedure for linear stability analysis we introduce a small perturbation $\delta U$ on the nonlinear bound state $U$. We plug

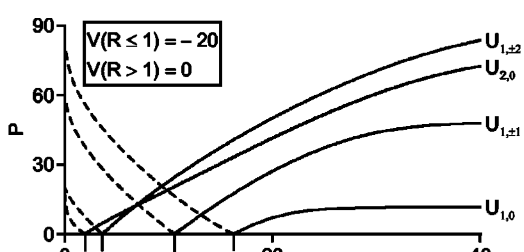


FIG. 2. "Soliton" power $P$ versus parameter $\beta$ ($\sigma=+1$ solid lines; $\sigma=-1$ dashed lines): The four discrete linear modes $\Theta_{j,M}$ can be seen as "solitons" with zero power.

$$\Psi = (U + \delta U)\exp(iM\Phi + i\beta Z) \tag{5}$$

into Eq. (2) and linearize with respect to the perturbation. The resulting evolution equation for the perturbation $\delta U$ is given by

$$i\frac{\partial}{\partial Z}\delta U - \beta\delta U - V\delta U + 2\sigma U^2\delta U + \sigma U^2\delta U^* + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)\delta U + \frac{1}{R^2}\left(\frac{\partial}{\partial \Phi} + iM\right)^2\delta U = 0. \tag{6}$$

With the ansatz

$$\delta U(R,\Phi,Z) = \delta U_1(R)\exp(im\Phi + i\lambda Z) + \delta U_2^*(R)\exp(-im\Phi - i\lambda^* Z), \tag{7}$$

we then derive the eigenvalue problem

$$\hat{L}\begin{pmatrix}\delta U_1\\ \delta U_2\end{pmatrix} = \lambda\begin{pmatrix}\delta U_1\\ \delta U_2\end{pmatrix}, \tag{8}$$

where $\delta U_1$ and $\delta U_2$ are independent complex functions and

$$\hat{L} = \begin{pmatrix}\hat{D}_{M+m} - \beta + 2\sigma U^2 & \sigma U^2\\ -\sigma U^2 & -\hat{D}_{M-m} + \beta - 2\sigma U^2\end{pmatrix}.$$

Since the resulting linear operator $\hat{L}$ is real, we expect pairs of eigenvalues $(\lambda, \lambda^*)$. If for a given bound state $U$ all eigenvalues $\lambda$ of Eq. (8) are real numbers, we call $U$ orbitally stable; otherwise, we call it linearly unstable.

Here we address the stability of these nonlinear bound states for low powers $P$. For $\sigma > 0$, this is the only regime where we can expect stability, because for high enough powers the linear potential in Eq. (2) becomes negligible, and all bound states of the resulting two-dimensional NLS equation are unstable, by either spreading or collapsing to a singularity [15]. So the question we want to answer is: does a linear mode $\Theta_{j,M}$ become always unstable if we increase the power or can it be continued into a nonlinear state $U_{j,M}$ remaining stable up to a certain threshold power? In order to shed light on this issue, we split the linear operator of the eigenvalue problem, Eq. (8), into two operators $\hat{L} = \hat{H} + \sigma\hat{N}$, where

$$\hat{H} = \begin{pmatrix}\hat{D}_{M+m} - \beta & 0\\ 0 & -\hat{D}_{M-m} + \beta\end{pmatrix} \tag{9}$$

is self-adjoint and

$$\hat{N} = \begin{pmatrix}2U^2 & U^2\\ -U^2 & -2U^2\end{pmatrix} \tag{10}$$

contains the dependence on the nonlinear bound state $U$. So for small powers $\hat{N}$ acts as a perturbation on $\hat{H}$. For $P = 0$ (or $\sigma = 0$) we have $\hat{L} = \hat{H}$, $\beta = \gamma_{j,M}$ and each row of Eq. (8) is equivalent to Eq. (3). Hence, in the spectrum of operator $\hat{H}$, all the linear modes of the waveguide structure are reproduced twice. The solutions of the eigenvalue problem

$$\hat{H}\begin{pmatrix}\delta U_1\\ \delta U_2\end{pmatrix} = \lambda\begin{pmatrix}\delta U_1\\ \delta U_2\end{pmatrix} \tag{11}$$

are the localized eigenfunctions (modes of the discrete spectrum of $\hat{H}$) defined as

$$\begin{pmatrix}\Theta_{k,M'}\\ 0\end{pmatrix}, \quad \lambda = \gamma_{k,M'} - \beta, \quad m = M' - M,$$

$$\begin{pmatrix}0\\ \Theta_{k,M'}\end{pmatrix}, \quad \lambda = -\gamma_{k,M'} + \beta, \quad m = M - M',$$

and the delocalized eigenfunctions (radiative modes of the continuous spectrum of $\hat{H}$) defined as

$$\begin{pmatrix}\Theta_{\gamma_c,M'}\\ 0\end{pmatrix}, \quad \lambda = \gamma_c - \beta, \quad m = M' - M,$$

$$\begin{pmatrix}0\\ \Theta_{\gamma_c,M'}\end{pmatrix}, \quad \lambda = -\gamma_c + \beta, \quad m = M - M',$$

where still $\beta = \gamma_{j,M}$. Note that the angular momentum $M'$ consists of the amount of both momenta $M$ (soliton) and $m$ (perturbation) fixed by our ansatz [Eqs. (5) and (7)].

If $\lambda \leqslant -\beta$ or $\lambda \geqslant \beta$, we always find a delocalized eigenfunction of Eq. (11), either

$$\begin{pmatrix}\Theta_{\gamma_c,M'}\\ 0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\ \Theta_{\gamma_c,M'}\end{pmatrix}.$$

Besides, for a certain range of potential depths, some of the discrete eigenvalues $\gamma_{k,M'}$ of Eq. (3) lie in the domain $2\beta - \gamma_{k,M'} < 0$, so that the *discrete* eigenvalues $\lambda = \pm(\gamma_{k,M'} - \beta)$ of the localized eigenfunctions

$$\begin{pmatrix}0\\ \Theta_{k,M'}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\Theta_{k,M'}\\ 0\end{pmatrix}$$

are embedded in the *continuous* parts of the spectrum. In such a configuration, there exists degeneration between a continuous and a discrete eigenvalue. This type of degeneration will be crucial throughout our analysis.

Before we go on and look at the effects of the perturbation $\hat{N}$, it might be helpful to illustrate these considerations with the example of Fig. 1. As mentioned above, with knowledge of the eigenvalues of Eq. (3) it is easy to construct the spectrum of the operator $\hat{H}$: each eigenvalue $\gamma$ in Eq. (3) creates a pair of eigenvalues $\lambda = \gamma - \beta$ and $\lambda = \beta - \gamma$ in the spectrum of $\hat{H}$. Figure 3 shows the construction of the spectrum for two different choices of $\beta$. For the vortex state with $\beta = \gamma_{1,\pm 1}$ [see Fig. 3(a)], the discrete eigenvalues and the two continua are clearly separated. The eigenvalues attached to the left axis belong to the first row of operator $\hat{H}$, while those appendant to the right axis belong to the second one. On the contrary, for the multihumped state with $\beta = \gamma_{2,0}$ [see Fig. 3(b)], some of the discrete eigenvalues are now embedded in the continuum. So formally we have a degeneration between continuous and discrete eigenvalues. This degeneration is

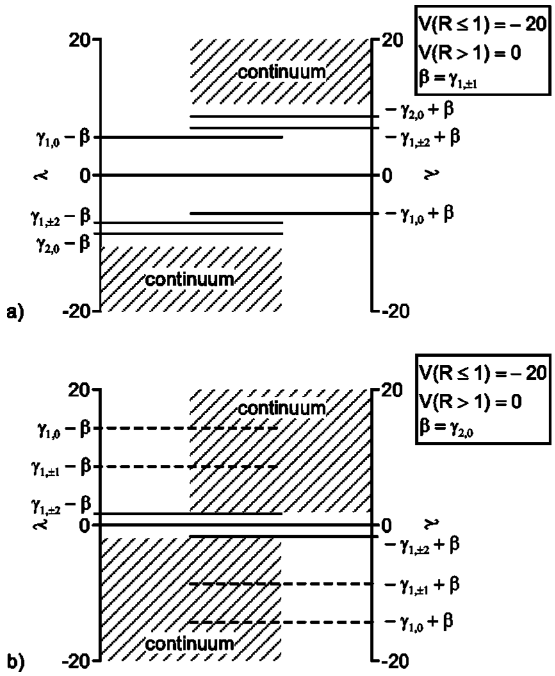


FIG. 3. Eigenvalues of Eq. (11) for two different choices of $\beta$. Two spectra of Fig. 1 are superimposed. One spectrum is shifted by an amount of $\beta$ downwards (left axis); the other one is mirrored ($\lambda \to -\lambda$) and shifted by an amount of $\beta$ upwards (right axis). In (a) the discrete and the continuous eigenvalues stay separated; in (b) we observe some discrete eigenvalues embedded in the continuum (dashed lines).

due to the fact that Eq. (11) is composed of two *decoupled* equations. When we introduce the perturbation $\hat{N}$, the latter property does no longer hold.

If we start with the spectrum of the operator $\hat{H}$ and switch on the perturbation $\hat{N}$, we expect that eigenvalues of $\hat{H}$ will shift. In particular, any degeneration between discrete eigenvalues and continuous eigenvalues, as described above, should be lifted. Note that trivially degenerated eigenvalues ($M' \to -M'$) will not split, since the generic symmetry is not changed by the nonlinearity. In contrast to $\hat{H}$, the operator $\hat{L}$ is not self-adjoint, so eigenvalues can become complex. But since $\hat{L}$ is real, this can only happen in pairs ($\lambda$, $\lambda^*$). *This property, therefore, implies that two eigenvalues have to be degenerated first, before the pair can move to the complex plane and destabilize the nonlinear bound state under consideration*. So, on the one hand, degeneration of two eigenvalues is necessary for destabilization. On the other hand, we know about any possible degeneration just from looking at the spectrum of the operator $\hat{H}$.

The same conclusion applies to more general nonlinearities and arbitrary dimensionality in Eq. (2), provided the equation features an "attractive" potential. The key point is that eigenvalues of the perturbation operator $\hat{L}$ only appear in pairs ($\lambda$, $\lambda^*$). An operator splitting $\hat{L}=\hat{H}+\sigma\hat{N}$ as above is always possible, where $\hat{H}$ is independent of the nonlinear bound state and self-adjoint. Then, our arguments using discrete eigenvalues of operator $\hat{H}$ embedded in the continuum

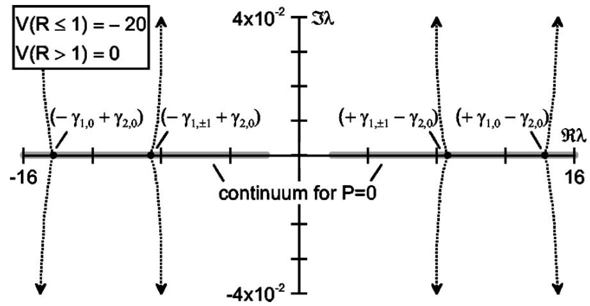


FIG. 4. The degeneracy of the embedded discrete eigenvalues is lifted by the cubic nonlinearity ($\sigma=+1$); the eigenvalues move to the complex $\lambda$ plane ($\Re\lambda \equiv \mathrm{Re}\lambda$, $\Im\lambda \equiv \mathrm{Im}\lambda$). The nonlinear bound state under consideration in this example is the multihumped state emanating from $\gamma_{2,0}$ ($\beta \geqslant \gamma_{2,0}$).

hold, because two eigenvalues of operator $\hat{L}$ have to be degenerated first, before the pair can move to the complex plane. The pairwise appearance of eigenvalues is, to our knowledge, generic for conservative, Hamiltonian systems.

## IV. STABILITY CRITERION FOR LOW-POWER BOUND STATES

The above arguments allow us to formulate a stability criterion for low-power nonlinear bound states of Eq. (2): If a certain localized linear mode $\Theta_{j_0,M_0}$ of Eq. (3) with eigenvalue $\gamma_{j_0,M_0}$ fulfills

$$\frac{1}{2}\gamma_{j,M} < \gamma_{j_0,M_0} \text{ for all } j,M, \tag{12}$$

the corresponding nonlinear bound state $U_{j_0,M_0}$ is linearly stable for small powers. On the contrary, if the criterion (12) is not fulfilled, an arbitrary small nonlinearity can lift the degeneracy in the spectrum of operator $\hat{L}$ by shifting the two eigenvalues to the complex plane and leading thereby to the instability of the nonlinear bound state. As a direct consequence of this criterion, the ground state with $\beta$ emanating from $\gamma_{1,0}$ is always stable. Figure 4 shows an illustrative example of this destabilization mechanism for $\sigma=+1$. Note that the criterion (12) applies to both focusing ($\sigma>0$) and defocusing ($\sigma<0$) nonlinearities.

Before proceeding further, it might be helpful to illustrate the criterion (12) in the picture of four-wave mixing. Here, we consider partially degenerate four-wave mixing: A strong "pump wave" with wave number $k_1$ creates two sidebands located symmetrically at wave numbers $k_3$ and $k_4$, obeying $k_1-k_3=k_4-k_1$, where we assume for definiteness $k_3<k_4$ (see, e.g., Ref. [16]). Of course, the creation of sidebands requires the existence of such waves in the medium. Brought forward to our system, the "pump wave" corresponds to the low-power nonlinear bound state $U_{j_0,M_0} \approx \Theta_{j_0,M_0}$ with $k_1=\beta \approx \gamma_{j_0,M_0}$. Another mode of the potential $\Theta_{j,M}$ can only be excited as a "third wave" ($k_3=\gamma_{j,M}$) if there is a matching "fourth wave", namely, $\Theta_{\gamma_c,M}$ with wave number $k_4=\gamma_c = 2\gamma_{j_0,M_0}-\gamma_{j,M}$. If the stability criterion (12) is fulfilled, no matching "fourth wave" exists. Hence, the "third wave" can-

not grow and cause instability of the nonlinear bound state $U_{j_0,M_0}$. On the contrary, if the criterion is not fulfilled, we always find a matching "fourth wave" in the continuum.

To clear up the effects of the nonlinearity, we consider the asymptotics of unstable eigenmodes (eigenfunctions of operator $\hat{L}$ with complex eigenvalue $\lambda$, $\mathrm{Im}\lambda<0$) with $\lim_{P\to 0}\mathrm{Re}\lambda \to \lambda_0 > \gamma_{j_0,M_0}$ and $0<|\mathrm{Im}\lambda| \ll \lambda_0$, $\mathrm{Im}\lambda<0$ (the case $\lambda_0 < -\gamma_{j_0,M_0}$ can be treated in a similar way). Since in the linear limit [Eq. (11)] all eigenmodes are stable, we have $\lim_{P\to 0}\mathrm{Im}\lambda \to 0$. By solving Eq. (8) in the asymptotic regime $R\to\infty$ and linearizing the exponential arguments with respect to $\mathrm{Im}\lambda$ we find

$$\delta U_1 \sim \frac{1}{\sqrt{R}} \exp\left[\left(-\sqrt{\mathrm{Re}\lambda+\beta} - i\frac{\mathrm{Im}\lambda}{2\sqrt{\mathrm{Re}\lambda+\beta}}\right)R\right] \quad (13)$$

$$\delta U_2 \sim \frac{1}{\sqrt{R}} \exp\left[\left(-i\sqrt{\mathrm{Re}\lambda-\beta} + \frac{\mathrm{Im}\lambda}{2\sqrt{\mathrm{Re}\lambda-\beta}}\right)R\right], \quad (14)$$

where both components $\delta U_1$ and $\delta U_2$ are localized (finite power integral). This is worth noticing, since in contrast to $\delta U_1$, the component $\delta U_2$ is delocalized in the linear limit $\mathrm{Im}\lambda=0$ ($P=0$ or $\sigma=0$). Due to the nondiagonal elements of the operator $\hat{N}$, we can conclude that both $\delta U_1$ and $\delta U_2$ have nonzero norm. More precisely, it is possible to show that

$$\mathrm{Im}\lambda \int (|\delta U_1|^2 - |\delta U_2|^2) R dR = 0 \quad (15)$$

(see the Appendix for details) and therefore

$$\int |\delta U_1|^2 R dR = \int |\delta U_2|^2 R dR \overset{!}{=} 1. \quad (16)$$

Henceforth the symbol $\overset{!}{=}$ signifies that the power integrals of $\delta U_1$ and $\delta U_2$ can be set equal to unity without loss of generality, by virtue of the linear nature of the equations which these perturbations satisfy. By doing so, Eq. (16) provides useful information on the maximum of their amplitude in the transverse plane. Together with the asymptotics [Eqs. (13) and (14)], we are now able to evaluate the dependency of $\max|\delta U_1|^2$ and $\max|\delta U_2|^2$ on $|\mathrm{Im}\lambda| \ll \lambda_0$, which will allow us to deduce further results.

### A. Dependence of $\max|\delta U_1|^2$ on $|\mathrm{Im}\lambda|$

Let us have a look at Eq. (13). Since the real part of the exponent $-\sqrt{\mathrm{Re}\lambda+\beta}R$ is independent of $\mathrm{Im}\lambda$, we can conclude that $\delta U_1(R)\approx 0$ at large distances $R \gg 1/\sqrt{\mathrm{Re}\lambda+\beta}$. Hence, the entire "mass" of $\delta U_1$ is concentrated at finite distances, where $U\neq 0$ and $V\neq 0$. Since $\int|\delta U_1|^2 R dR \overset{!}{=} 1$, we have thus

$$\max|\delta U_1|^2 \approx C, \quad (17)$$

where $C$ is a *nonzero* constant independent of $\mathrm{Im}\lambda$.

Equation (17) implies that the criterion (12) is indeed *sufficient* for small powers: For a vanishing nonlinearity, we have $\mathrm{Im}\lambda=0$. For continuity reasons, these unstable eigenmodes must converge to a *localized* eigenfunction of the operator $\hat{H}$ with discrete eigenvalue embedded in the continuum. Otherwise, $\max|\delta U_1|^2$ would jump from $C\neq 0$ to zero for power $P\to 0$, because delocalized eigenfunctions cannot keep a finite power integral while having nonzero amplitude. Conversely, since the perturbations become localized in one component ($\delta U_1$) and delocalized in the other one ($\delta U_2$) in the limit $P\to 0$, instability at low powers *necessarily* starts from discrete eigenvalues "embedded" in the continuum of the operator $\hat{H}$. Furthermore, with this knowledge, we can specify the above constants $\lambda_0=\gamma_{j,M}-\gamma_{j_0,M_0}$ and $C=\max|\Theta_{j,M}|^2$ with $\int|\Theta_{j,M}|^2 R dR \overset{!}{=} 1$.

### B. Dependence of $\max|\delta U_2|^2$ on $|\mathrm{Im}\lambda|$

For $|\mathrm{Im}\lambda| \ll \lambda_0$ all the mass of $\delta U_2$ lies at large distances $R\to\infty$, and with Eq. (14) we are able to compute $\int|\delta U_2|^2 R dR \sim 1/|\mathrm{Im}\lambda|$. Since we have fixed $\int|\delta U_2|^2 R dR \overset{!}{=} 1$, the previous estimate implies that

$$\max|\delta U_2|^2 \sim |\mathrm{Im}\lambda|. \quad (18)$$

Besides, if we remember that Eq. (8) is a linear differential equation, it is obvious that there exists a Green's function $G(R,R')$ with

$$\delta U_2(R) = \int G(R,R')\sigma U(R')^2 \delta U_1(R') R' dR', \quad (19)$$

which means

$$\max|\delta U_2| \sim \sigma \max U^2. \quad (20)$$

Hence, we obtain

$$\sqrt{|\mathrm{Im}\lambda|} \sim \sigma \max U^2, \quad (21)$$

so the growth rate $|\mathrm{Im}\lambda|$ of the unstable eigenmode is proportional to the squared nonlinearity.

In order to express this dependency in terms of the "soliton parameter" $\beta$, we may perform a perturbative analysis in the limit $|\sigma| \ll 1$, similarly to Refs. [9,10]. We expand the nonlinear bound state $U$ as

$$U = \Theta + \sigma U^{(1)} + \mathcal{O}(\sigma^2) \quad (22)$$

and

$$\beta = \gamma + \sigma\beta^{(1)} + \mathcal{O}(\sigma^2), \quad (23)$$

where $\Theta \equiv \Theta_{j_0,M_0}$ and $\gamma \equiv \gamma_{j_0,M_0}$ satisfy Eq. (3). Plugging the ansatz $\psi = U\exp(iM_0\Phi + i\beta Z)$ into Eq. (2), it is readily found that

$$\beta^{(1)} = \frac{\int \Theta^4 R dR}{\int \Theta^2 R dR} \sim \max \Theta^2. \quad (24)$$

Hence, with Eq. (23) we have

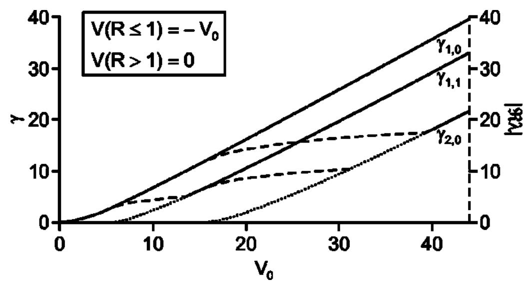


FIG. 5. Eigenvalue $\gamma$ versus the depth of the step potential $V$ for the linear modes $\Theta_{1,0}$, $\Theta_{1,1}$, and $\Theta_{2,0}$. Solid lines indicate a free-of-degeneration spectrum of operator $\hat{H}$; domains with discrete eigenvalues embedded in the continuum are specified in dotted lines. The dashed lines show the absolute values of the embedded eigenvalues $\lambda$ of Eq. (11).

$$\beta - \gamma \sim \sigma \max \Theta^2. \tag{25}$$

For small $|\sigma|$, thus $\max \Theta^2 \sim \max U^2$ [see Eq. (22)], the nonlinearity depends on the "soliton parameter" as

$$\sigma \max U^2 \sim \beta - \gamma. \tag{26}$$

Combining Eqs. (21) and (26) we get

$$|\mathrm{Im}\lambda| \sim (\beta - \gamma)^2. \tag{27}$$

The need that eigenvalues of operator $\hat{L}$ have to be degenerated first before moving to the complex plane and cause

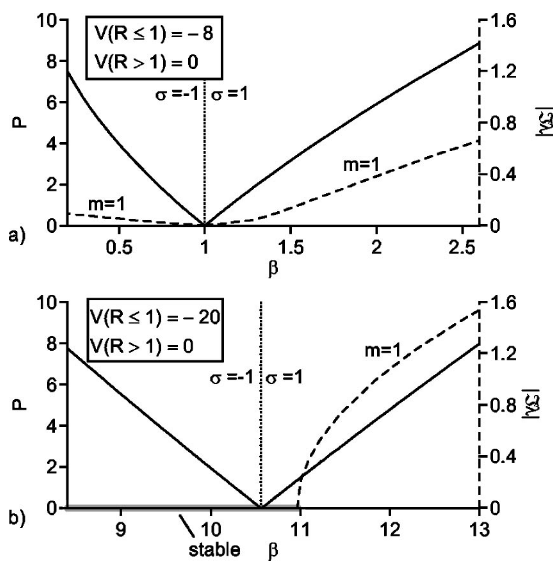


FIG. 6. Soliton power $P$ versus soliton parameter $\beta$ for the vortex state $M=1$ (solid line). The dashed line shows the computed growth rate $|\mathrm{Im}\lambda|$ of the unstable eigenmode. If (a) $\frac{1}{2}\gamma_{1,0}-\gamma_{1,1}\geqslant 0$, the vortex nonlinear bound state is unstable, whereas with (b) $\frac{1}{2}\gamma_{1,0}-\gamma_{1,1}\leqslant 0$ it is stable for small powers.

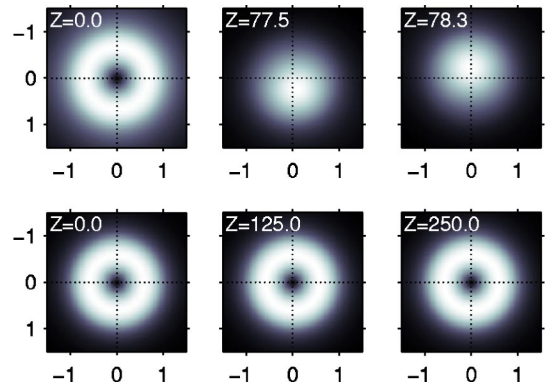


FIG. 7. The vortex nonlinear bound state $M=1$, $\sigma=1$: instability for $\beta=2.3$ and $V_0=8$ (row I); and stability for $\beta=11$ and $V_0=20$ (row II).

instability yields a sufficient condition for stability of low-power nonlinear bound states in the present framework. However, determining compellable analytical arguments showing that the embedded discrete eigenvalues of the operator $\hat{H}$ *always* lead to complex eigenvalues for the operator $\hat{L}$ (unstable eigenmodes) is still an open issue. Nevertheless, in the numerical examples discussed below we always observe this destabilization mechanism.

## V. NUMERICAL RESULTS

Let us return to our example, the radial step potential, and check the above theoretical predictions. We will concentrate on the "solitons" emanating from $\gamma_{1,1}$ (vortex state) and from $\gamma_{2,0}$ (multihumped state) (see Fig. 2). Figure 5 shows the regions of degenerated $\lambda$ of Eq. (11) for the three linear modes $\Theta_{1,0}$, $\Theta_{1,1}$, and $\Theta_{2,0}$. Up to a certain depth of the confining potential $V$, the stability criterion (12) is not fulfilled for the higher-order modes. At this depth, the last "embedded eigenvalue" leaves the continuum. In this context it is important to point out that the above destabilization process is *not* due to weak linear guiding of the respective "soliton." If this were the case, we would observe a certain nonzero critical power, below which the "soliton" would be stable. In contrast to this scenario, the destabilization appears for *arbitrary small* power.

Since the degeneration of the eigenvalues of Eq. (11) involves the continuum, the related unstable eigenmodes are stretched over a large area, especially for very small powers. Therefore, conventional solver for Eq. (8), like, e.g., the NAG routine F02ECF [17], failed due to the necessity of a very large computational window. We worked around this problem in solving Eq. (6) directly with a beam-propagation method (Crank-Nicholson) and transparent boundary conditions.

To confirm the results of our stability analyses, we present full 2D simulations illustrating the decay of unstable low-power vortex and multihumped "solitons." It turns out that, at least for the examples presented here, the final state is correlated with the dominant unstable eigenmode (here the ground state).

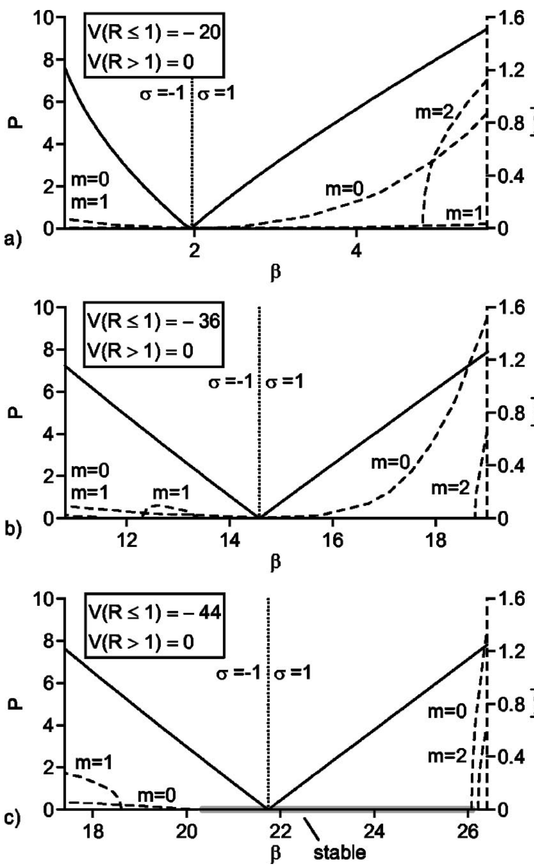


FIG. 8. Soliton power $P$ versus parameter $\beta$ for the multihumped state $M=0$ (solid line). The dashed lines show the computed growth rates $|\mathrm{Im}\lambda|$ of the unstable eigenmodes. If (a) $\frac{1}{2}\gamma_{2,0}-\gamma_{1,1}\geqslant 0$ and $\frac{1}{2}\gamma_{2,0}-\gamma_{1,0}\geqslant 0$, the nonlinear bound state has two unstable eigenmodes, whereas with (b) $\frac{1}{2}\gamma_{2,0}-\gamma_{1,1}\leqslant 0$ and $\frac{1}{2}\gamma_{2,0}-\gamma_{1,0}\geqslant 0$, it has one unstable eigenmode ($m=0$). In the case of (c) $\frac{1}{2}\gamma_{2,0}-\gamma_{1,1}\leqslant 0$ and $\frac{1}{2}\gamma_{2,0}-\gamma_{1,0}\leqslant 0$, the multihumped state is stable for small powers.

### A. Vortex state

We first discuss the vortex state. If $6<V_0<13.6$ (boundaries here correspond to the lower and upper bounds in Fig. 5 for eigenvalue degeneracy in the limit $P\rightarrow 0$), we find $\frac{1}{2}\gamma_{1,0}>\gamma_{1,1}$. Hence, the degenerated eigenvalues of Eq. (11) are $\pm(\gamma_{1,0}-\gamma_{1,1})$. So we can guess that the low-power nonlinear vortex state in this range of $V_0$ has an unstable eigenmode with $m=\pm 1$, so that either $\lim_{P\rightarrow 0}\delta U_1=\Theta_{1,0}$ ($M+m=0$) or $\lim_{P\rightarrow 0}\delta U_2=\Theta_{1,0}$ ($M-m=0$). The real part of this complex eigenvalue is approximately $|\mathrm{Re}\lambda|\approx\gamma_{1,0}-\gamma_{1,1}$. With a deeper potential than the critical value $V_0=13.6$, this instability is expected to disappear. In order to check these predictions, a numerical stability analysis of the vortex state was performed for $V_0=8$ and $V_0=20$. As expected, for $V_0=20$ we observed a stable region of the vortex branch for small powers, whereas for $V_0=8$ the branch becomes immediately unstable when the nonlinearity comes into play (see Fig. 6). We successfully checked numerically the "instability onset value" for the depth of the potential $V_0=13.6$. The "instability onset value" and the value at which the first discrete eigenvalue just touches the continuum were observed to coincide.

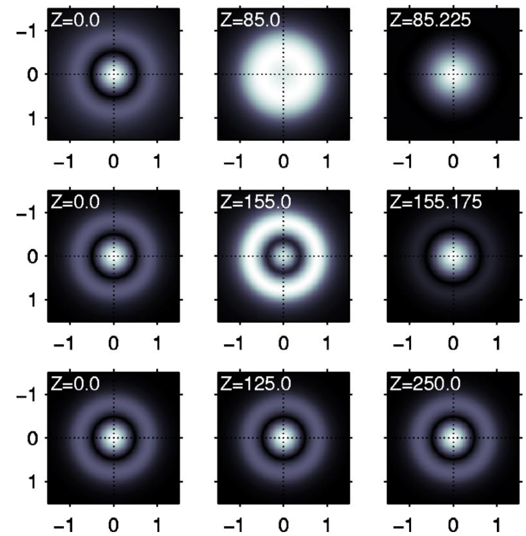


FIG. 9. The multihumped nonlinear bound state $M=0$, $\sigma=1$: instability for $\beta=4.3$ and $V_0=20$ (row I); instability for $\beta=16$ and $V_0=36$ (row II); and stability for $\beta=24$ and $V_0=44$ (row III).

As far as direct simulations are concerned, the first row in Fig. 7 shows the decay of the vortex "soliton" with $\beta=2.3$ from the potential depth selected in Fig. 6(a). Perturbed with 1% random amplitude noise at $Z=0$, the growing unstable eigenmode ($m=1$) destroys the "doughnut shape" of the vortex. At $Z=77.5$ and $Z=78.3$, respectively, we show two snapshots of the asymptotic behavior, which consists in a spinning single-hump solution with a period $\Delta Z\approx 1.7$. This single hump is nothing else but the nonlinear ground state $U_{1,0}$, where additional rotation is induced by a stable eigenmode with $m=1$ and $\lambda=2\pi/\Delta Z\approx 3.7$. Because we are in the low-power regime, we can identify the connection of this eigenmode with $\Theta_{1,1}$, the linear vortex state in Eq. (3). Indeed, for $V_0=8$ we find $\lambda\approx 3.7\approx\gamma_{1,0}-\gamma_{1,1}$ (see also Fig. 5). The second row of Fig. 7 displays the numerical verification of the stability predicted for the vortex state with $V_0=20$.

### B. Multihumped state

In the case of the multihumped state, Fig. 5 shows that, for $16<V_0<31.2$, Eq. (11) features four degenerated eigenvalues $\pm(\gamma_{1,0}-\gamma_{2,0})$ related to the ground state $\Theta_{1,0}$, and $\pm(\gamma_{1,1}-\gamma_{2,0})$ related to the vortex state $\Theta_{1,1}$. For potentials deeper than $V_0=31.2$ the degeneration of the vortex state first vanishes, and with $V_0>39.2$ the stability criterion (12) becomes fulfilled. Again, we can confirm these behaviors with a numerical stability analysis (see Fig. 8). For $V_0=20$ the multihumped branch is unstable for arbitrary small powers due to two unstable eigenmodes. The eigenmode linked to the ground state ($m=0$) possesses a substantially larger growth rate, so we can expect this one to play the crucial role in the decay of the multihumped nonlinear bound state (see full 2D simulations below). The unstable eigenmode associated with the vortex state disappears for a deeper potential ($V_0=36$), and choosing $V_0=44$ we observe stability for a quite large range of soliton power.

The decay of the multihumped “solitons” produced by the potentials used in Figs. 8(a) and 8(b) with $\beta=4.3$ and $\beta=16$ is shown in row I and row II of Fig. 9, respectively. Again we end up with the stable ground-state breathing due to stable eigenmodes with $m=0$. The periods of oscillation, $\Delta Z\approx 0.45$ and $\Delta Z\approx 0.35$, between the two extremal beam shapes shown in the respective last two pictures of the rows are compatible with the eigenvalues $\lambda=14$ and $\lambda=18$: we retrieve these values as the difference $\gamma_{1,0}-\gamma_{2,0}$ computable from Fig. 5. As expected, a sufficient depth ($V_0=44$) of the potential stabilizes the higher-order nonlinear bound state with $M=0$ (last row).

## VI. CONCLUSION

In summary, we have presented a sufficient stability criterion for weakly nonlinear bound states, which only involves the linear eigenvalue problem ($P\to 0$ or $\sigma=0$). Simple knowledge of the eigenvalues associated with the linear modes of the potential $V$ allows us to predict the stability of the nonlinear bound states of the extended NLS equation (2). In spite of the fact that the criterion is valid for low-power solitons only, the example of a step potential shows that the present results may hold for wider ranges of power, both for focusing and defocusing nonlinearities.

## APPENDIX: COMPONENTS OF UNSTABLE MODES HAVE EQUAL NORM

We prove the relation $\mathrm{Im}\lambda\int(|\delta U_1|^2-|\delta U_2|^2)RdR=0$ [Eq. (15)], using the notations $a=\delta U_1$, $b=\delta U_2$, $I_0=\sigma U^2$, and $\lambda=\lambda'+i\lambda''$ for convenience:

$$\lambda''\int(|a|^2-|b|^2)RdR=\frac{1}{2i}\int\{[a^*i\lambda''a-\text{c.c.}]-[b^*i\lambda''b-\text{c.c.}]\}RdR$$

$$=\frac{1}{2i}\int\{[a^*(\hat{D}_{M+m}-\beta-\lambda'+2I_0)a+a^*I_0b-\text{c.c.}]-[b^*(-\hat{D}_{M-m}+\beta-\lambda'-2I_0)b-b^*I_0a-\text{c.c.}]\}RdR$$

$$=\frac{1}{2i}\int\left\{\left[a^*\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)a+a^*I_0b-\text{c.c.}\right]+\left[b^*\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)b+b^*I_0a-\text{c.c.}\right]\right\}RdR,$$

where we used Eq. (8) and the fact that the quantities $\lambda'$, $\beta$, $V$, $I_0$, and $M^2/R^2$ are real valued. In the above equation, it is easy to see that $[a^*I_0b-\text{c.c.}]+[b^*I_0a-\text{c.c.}]=0$, which leaves us with the task to show that

$$\int\left[f^*\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)f-f\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)f^*\right]dR=0,$$

where either $f=a$ or $f=b$. Since $\lim_{R\to\infty}f,\partial_R f=0$, integration by parts in both terms immediately shows the desired result.

Similar integral relations were previously established in [12], in order to prove that a resonance of two localized eigenmodes produce soliton instability in the context of the parametrically driven NLS equation.


[1] A. C. Newell and J. V. Moloney, *Nonlinear Optics* (Addison-Wesley, Reading, MA, 1992).
[2] H. A. Rose and M. I. Weinstein, Physica D **30**, 207 (1988).
[3] T. J. Alexander and L. Bergé, Phys. Rev. E **65**, 026611 (2002); see also L. Bergé, *ibid.* **62**, R3071 (2000).
[4] J. Schjødt-Eriksen, Yu. B. Gaididei, and P. L. Christiansen, Phys. Rev. E **64**, 066614 (2001).
[5] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[6] C. A. Sackett *et al.*, Phys. Rev. Lett. **82**, 876 (1999); J. M. Gerton *et al.*, Nature (London) **408**, 692 (2000); E. A. Donley *et al.*, *ibid.* **412**, 295 (2001).
[7] M. R. Matthews *et al.*, Phys. Rev. Lett. **83**, 2498 (1999).
[8] Yu. S. Kivshar, T. J. Alexander, and S. K. Turitsyn, Phys. Lett. A **278**, 225 (2001).
[9] D. L. Feder, M. S. Pindzola, L. A. Collins, B. I. Schneider, and C. W. Clark, Phys. Rev. A **62**, 053606 (2000).
[10] D. V. Skryabin, Phys. Rev. A **63**, 013602 (2001).
[11] I. V. Barashenkov, M. M. Bogdan, and V. I. Korobov, Europhys. Lett. **15**, 113 (1991).
[12] N. V. Alexeeva, I. V. Barashenkov, and D. E. Pelinovsky, Nonlinearity **12**, 103 (1999).
[13] V. I. Korneev, N. N. Akhmediev, and T. D. Shermergor, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **37**, 1525 (1984).
[14] A. W. Snyder, D. J. Mitchell, and L. Poladian, J. Opt. Soc. Am. B **8**, 1618 (1991).
[15] J . J. Rasmussen and K. Rypdal, Phys. Scr. **33**, 481 (1986).
[16] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, San Diego, 1989).
[17] G. H. Golub and C. F. van Loan, *Matrix Computations* (Johns Hopkins University Press, Baltimore, 1996).